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Symmetry bounds of variational problems

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Abstract. Sharp upper bounds on the dimension of the Lie algebra of infinitesimal variational and divergence point symmetries of a non-trivial Lagrangian $L(x, u, u', \ldots, u^{(n)})(x, u \in \mathbb{R})$ of arbitrary order n are found. For any given order, all Lagrangians whose Lie algebra of variational or of divergence symmetries is of maximal dimension are completely classified, modulo local point transformations. It is shown, in particular, that for $n \geqslant 2$ the algebra of variational symmetries of the generalized free particle Lagrangian $(u^{(n)})^2$ is *not* of maximal dimension, whereas when $n = 1$ there are several Lagrangians admitting a variational symmetry algebra of maximal dimension and generating differential equations different from the free particle equation. A connection between variational problems on the line and scalar evolution equations in one time and one space variables is also established, showing that Lagrangians with a variational symmetry algebra of maximal dimension correspond to evolution equations with a maximal Lie algebra of time-preserving time-independent infinitesimal point symmetries. The technique used in the proof of the above results is applied to give a simple proof of the fact that an ordinary differential equation of order $n > 2$ has a symmetry algebra of maximal dimension if and only if it is locally equivalent under a point transformation to the generalized free particle equation $u^{(n)} = 0$.

1. Introduction

One of the first results obtained by Sophus Lie in his development of group theory is the fact that the Lie algebra of vector fields

$$X = \xi(x, u)\partial_x + \eta(x, u)\partial_u \tag{1.1}$$

generating point symmetry transformations (*symmetry algebra* for short) of an ordinary differential equation

$$u^{(k)} = f(x, u, u', \ldots, u^{(k-1)}) \tag{1.2}$$

of order $k > 1$, is finite-dimensional, provided that the components of the vector field X are real analytic. In fact, Lie [1] showed that the dimension of this algebra is bounded above by the integers

$$\begin{cases} 8 & \text{if } k = 2 \\ k + 4 & \text{if } k > 2. \end{cases} \tag{1.3}$$

(This result has been extended recently to the case $X \in C^\infty$, cf [2].) Moreover, Lie also proved that these upper bounds are sharp, i.e. for every $k > 1$ there is a kth-order differential

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equation (1.2) whose symmetry algebra has dimension exactly equal to (1.3). The simplest example of an equation having such a property is given for all $k \geqslant 2$ by

$$u^{(k)} = 0 \tag{1.4}$$

which we shall call the *free particle equation* by analogy with the $k = 2$ case.

The above results have been partially generalized to systems of differential equations in normal form, i.e. such that each equation in the system can be solved for the highest-order derivative appearing in it. More precisely, it has been shown in [3] that the Lie algebra of (real analytic) infinitesimal point symmetries of a system of ordinary differential equations in normal form containing no first-order equations is always finite-dimensional. Moreover, in the case of a normal system of m equations of order k

$$u_i^{(k)} = f_i(x, u_1, \ldots, u_m, \ldots, u_1^{(k-1)}, \ldots, u_m^{(k-1)}) \qquad i = 1, 2, \ldots, m \tag{1.5}$$

an upper bound on the dimension of the symmetry algebra has also been found in [3] and later refined in [4] (see also [5]), namely

$$\begin{cases} m^2 + 4m + 3 & \text{if } k = 2 \\ m^2 + m(k+1) + 2 & \text{if } k > 2. \end{cases} \tag{1.6}$$

However, this upper bound is known to be sharp only for $k = 2$, when it is achieved by the m-dimensional free particle equation

$$u_i'' = 0 \qquad i = 1, 2, \ldots, m.$$

The symmetry algebra of the m-dimensional generalized free particle equation of order $k > 2$

$$u_i^{(k)} = 0 \qquad i = 1, 2, \ldots, m$$

has also been computed in [3], but its dimension $m^2 + km + 3$ is strictly less than (1.6) when $k > 2$. Therefore, for $k > 2$ it is not known at present whether or not, for a given order k, the m-dimensional free particle equation of order k is the m-dimensional system with the largest symmetry algebra.

Similar results for time-independent time-preserving symmetries of evolution equations were recently obtained by Sokolov, [6]. Indeed, the Lie algebra of smooth (but not necessarily real analytic) infinitesimal point symmetries of the form

$$0 \cdot \partial_t + \xi(x, u)\partial_x + \eta(x, u)\partial_u \tag{1.7}$$

of an evolution equation

$$u_t = f(x, u, u_1, \ldots, u_n) \qquad u_i \equiv \frac{\partial^i u}{\partial x^i} \tag{1.8}$$

is of $n + 3$ dimension at most when $n > 1$, and this bound is optimal. (Actually, Sokolov's result is slightly more general, since it applies to *contact* symmetries as well, cf [6].)

Surprisingly, the analogous problems for ordinary differential equations (1.2) arising as Euler–Lagrange equations of a Lagrangian $L(x, u, u', \ldots, u^{(n)})$ have received little attention in the literature. In this context, the 'natural' point symmetry transformations are of two types, variational and divergence, whose definition we shall now briefly recall.

Point transformations that preserve the *action*

$$S_L[f] = \int_{x_0}^{x_1} L(x, f(x), f'(x), \ldots, f^{(n)}(x))\,\mathrm{d}x \tag{1.9}$$

associated to the Lagrangian L are called *variational symmetries* of L. A vector field (1.1) is an *infinitesimal variational symmetry* of L if it generates a one-parameter group of variational symmetries of L. The necessary and sufficient condition for a vector field (1.1) to be an infinitesimal variational symmetry of a Lagrangian L is well known, namely

$$\mathrm{pr}^{(n)} X \cdot L + L D_x \xi = 0. \tag{1.10}$$

The notation used in this formula is that of reference [7], i.e. D_x is the *total derivative operator* defined by the formally infinite sum

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} \tag{1.11}$$

with

$$u_i \equiv u^{(i)} \tag{1.12}$$

and

$$\mathrm{pr}^{(n)} X = X + \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial u_i} \tag{1.13}$$

is the *nth prolongation* of the vector field X, whose components $\eta^i(x, u, \ldots, u_i)$ are given by the usual formulas

$$\eta^i = D_x^i(\eta - u_x\xi) + u_{i+1}\xi \qquad i = 1, 2, \ldots. \tag{1.14}$$

Divergence symmetries are usually defined directly at the infinitesimal level, cf [7], by a natural generalization of (1.10). More precisely, a vector field (1.1) is an *infinitesimal divergence symmetry* of a Lagrangian $L(x, u, \ldots, u_n)$ if it satisfies

$$\mathrm{pr}^{(n)} X \cdot L + L D_x \xi = D_x f \tag{1.15}$$

for some function $f(x, u, \ldots, u_{n-1})$. Roughly speaking, this means that the one-parameter group generated by (1.1) preserves the action (1.9) up to the addition of a boundary term.

The analytic characterizations (1.10) and (1.15) of the infinitesimal variational and divergence symmetries of a Lagrangian L are easy to understand if we consider the standard *contact structure* on the jet bundle $J^n(\mathbb{R}, \mathbb{R}) \equiv J^n$ with local coordinates $(x, u, u_1, \ldots, u_n)$. To this end, we introduce the usual basis of contact 1-forms $C_n = \{\theta^1, \ldots, \theta^n\}$, defined by

$$\theta^i = \mathrm{d}u_{i-1} - u_i\,\mathrm{d}x \qquad i = 1, 2, \ldots, n. \tag{1.16}$$

The necessary and sufficient condition (1.10) for the vector field (1.1) to be an infinitesimal variational symmetry of the Lagrangian $L(x, u, \ldots, u_n)$ is then simply expressed by the equation

$$\mathcal{L}_{\mathrm{pr}^{(n)}X}(L\,\mathrm{d}x) = 0 \bmod \mathcal{C}_n \tag{1.17}$$

whereas infinitesimal divergence symmetries of L satisfy

$$\mathcal{L}_{\mathrm{pr}^{(n)}X}(L\,\mathrm{d}x) = \mathrm{d}f \bmod \mathcal{C}_n \tag{1.18}$$

with $\mathcal{L}_V\omega$ denoting the Lie derivative of the differential form ω along the vector field V.

The Euler–Lagrange equation

$$\sum_{i=0}^{n}(-D_x)^i\frac{\partial L}{\partial u_i} = 0 \tag{1.19}$$

generated by a Lagrangian $L(x, u, \ldots, u_n)$ can always be locally expressed as a normal ordinary differential equation of order $2r$ for an appropriate r, with $0 \leqslant r \leqslant n$. For example, if L is *non-degenerate*, i.e. if $\partial^2 L/\partial u_n^2$ does not vanish identically, then the Euler–Lagrange equation is a normal differential equation of order $2n$ on a certain open subset, and therefore $r = n$ in this case. On the opposite end, when L can be written as

$$L = L_0(x, u) + D_x f \tag{1.20}$$

for some function $f : J^{n-1} \to \mathbb{R}$, then $r = 0$, since the Euler–Lagrange equation of L reduces to the trivial constraint

$$\frac{\partial L_0}{\partial u}(x, u) = 0.$$

We shall call a Lagrangian of the form (1.20) a *trivial Lagrangian*; a particular instance of such trivial Lagrangians are *null Lagrangians*, of the form $L = D_x f$. In this paper we shall deal exclusively with non-trivial Lagrangians, whose Euler–Lagrange equations are genuine differential equations. If L is a non-trivial C^∞ Lagrangian, its Lie algebras of C^∞ infinitesimal variational and divergence symmetries need not be finite-dimensional. For instance, if L vanishes for (x, u) outside an open subset M then any C^∞ vector field vanishing on an open subset strictly containing M is an infinitesimal variational symmetry of L, by (1.10) and the prolongation formula (1.14). On the other hand, if $L(x, u, \ldots, u_n)$ is C^∞ (or even merely of class C^{n+1}, so that its Euler–Lagrange equation (1.19) is well defined) and non-trivial, then its Lie algebras of *real analytic* infinitesimal variational and *divergence symemtries are finite-dimensional. Indeed, the Euler–Lagrange equation of* L is a normal ordinary differential equation on some open subset of J^{2r} for appropriate r, and therefore its symmetry algebra $\mathfrak{g}$ is a finite-dimensional Lie algebra of vector fields defined on some open subset $M \subset \mathbb{R}^2$. Hence the restriction to M of $\mathfrak{g}^{\mathrm{div}}$—the Lie algebra of real analytic infinitesimal divergence symmetries of L—is finite-dimensional, being a subalgebra of $\mathfrak{g}$. That is easily seen to imply that $\mathfrak{g}^{\mathrm{div}}$ itself is finite-dimensional (and $\dim\mathfrak{g}^{\mathrm{div}} \leqslant \dim\mathfrak{g}$), using the well known result that a real analytic function vanishing on a non-empty open subset must be identically zero on its whole domain. We shall therefore assume from now on that all Lagrangians are non-trivial and of class C^∞, and we shall only be interested in real analytic infinitesimal variational or divergence symmetries thereof.

Under these assumptions, both the variational and the divergence symmetry algebras are then finite-dimensional.

Once the finite dimension of the Lie algebras of variational and of divergence symmetries of any non-trivial Lagrangian has been established (under the above hypotheses), we are faced with problems totally analogous to those considered above for ordinary differential equations in normal form and for evolution equations. First of all, we would like to compute a sharp upper bound on the dimension of the variational and divergence symmetry algebras of any Lagrangian $L(x, u, \ldots, u_n)$ as a function of its order $n \geqslant 1$. Once this is done, it is also of interest to classify all Lagrangians of a given order n whose Lie algebra of variational or of divergence symmetries is of maximal dimension for that order. These are precisely the two main goals we have set outselves in this paper.

To perform a classification of maximally symmetric Lagrangians, an appropriate equivalence relation has to be specified. However, as we shall discuss next, it is natural to use slightly different equivalence relations in the variational and in the divergence cases. To begin with, if we perform a local change of variables

$$(x, u) \mapsto (\bar{x}, \bar{u}) = (\varphi(x, u), \psi(x, u)) = \Phi(x, u) \tag{1.21}$$

it is well known that a Lagrangian $L(x, u, \ldots, u_n)$ is expressed in the new local coordinates $(\bar{x}, \bar{u}, \ldots, \bar{u}_n)$ of J^n by the function $\bar{L}(\bar{x}, \bar{u}, \ldots, \bar{u}_n)$ related to L by

$$L = (\bar{L} \circ \mathrm{pr}^{(n)}\, \Phi) D_x \bar{x} \tag{1.22}$$

where the natural *prolongation* $\mathrm{pr}^{(n)}\, \Phi$ of (1.21) to J^n is given by

$$\mathrm{pr}^{(n)}\, \Phi(x, u, \ldots, u_n) = (\bar{x}, \bar{u}, \ldots, \bar{u}_n) \tag{1.23}$$

with

$$\bar{u}_i = \left(\frac{1}{D_x \varphi} D_x\right)^i \psi \qquad 1 \leqslant i \leqslant n. \tag{1.24}$$

It is clear that (1.22) preserves the variational symmetry algebra and the Euler–Lagrange equation, by the well known invariance of these concepts under point transformations, cf [7]. We therefore make the following definition:

Definition 1.1. Two Lagrangians $L(x, u, \ldots, u_n)$ and $\bar{L}(\bar{x}, \bar{u}, \ldots, \bar{u}_n)$ are *equivalent* if there is a change of variables (1.21) and a non-zero constant c such that L and $\bar{L}$ are related by

$$L = c(\bar{L} \circ \mathrm{pr}^{(n)}\, \Phi) D_x \bar{x}. \tag{1.25}$$

(Multiplication by a constant factor obviously does not change the variational symmetry algebra nor the Euler–Lagrange equation, and its inclusion in the previous definition is desirable in that it allows us to choose any convenient normalizaiton for L in each equivalence class.)

It is this notion of equivalence that we shall use to classify all Lagrangians possessing a Lie algebra of infinitesimal variational symmetries of maximal dimension. On the other hand, for the divergence symmetry classification it is more natural to use the following generalization of (1.25):

Definition 1.2. Two Lagrangians $L(x,u,\ldots,u_n)$ and $\bar{L}(\bar{x},\bar{u},\ldots,\bar{u}_n)$ are *divergence equivalent* if they are equivalent up to the total derivative of a function $f : J^{n-1} \to \mathbb{R}$, i.e. if they satisfy

$$L = c(\bar{L} \circ \mathrm{pr}^{(n)}\, \Phi) D_x \bar{x} + D_x \cdot f(x,u,\ldots,u_{n-1}). \tag{1.26}$$

Notice that (1.26) preserves the divergence symmetry algebra (and the Euler–Lagrange equation), but in general it maps infinitesimal variational symmetries of L into infinitesimal *divergence* symmetries of $\bar{L}$. That is the reason why we cannot use the more general equivalence relation (1.26) for both the variational and the divergence classification.

The paper is organized as follows. In section 2, we shall study the variational symmetry algebra of non-trivial Lagrangians of order n, showing that its dimension is at most $n+3$, if $n \geqslant 2$, or 3, if $n = 1$. We shall also prove that this upper bound is sharp, and we shall locally classify all Lagrangians with a variational symmetry algebra of maximal dimension. These results are a bit surprising, since the dimension of the symmetry algebra of the *free particle Lagrangian*

$$L = u_n^2 \tag{1.27}$$

whose Euler–Lagrange equation is the free particle equation (1.4) of order $2n$, is only $n+2$ when $n \geqslant 2$. Thus, contrary to what the case of scalar ordinary differential equations might suggest, the free particle Lagrangian does not have a maximal Lie algebra of infinitesimal variational symmetries if $n \geqslant 2$. If $n = 1$, the free particle Lagrangian does admit a variational symmetry algebra of maximal dimension, but as we shall see there are several other inequivalent first-order Lagrangians with this property. Section 3 is devoted to the same problems for infinitesimal divergence symmetries. We shall show that the divergence symmetry algebra of a non-trivial Lagrangian of order n is of dimension not greater than $2n+3$, and that any Lagrangian whose divergence symmetry algebra is $(2n+3)$-dimensional is locally divergence equivalent to the free particle Lagrangian (1.27). Finally, in section 4 we shall briefly explore some applications of the previous results to ordinary differential equations and evolution equations.

2. Variational symmetries

In this section we study the dimension of the Lie algebra of (real analytic) infinitesimal variational symmetries of a non-trivial Lagrangian $L(x,u,\ldots,u_n)$ of order $n \geqslant 1$. Since the free particle Lagrangian (1.27) generates the free particle equation (1.4) (with $k = 2n$), which has an infinitesimal symmetry algebra of maximal dimension, it would be natural to conjecture that (1.27) admits a variational symmetry algebra of maximal dimension. We shall see below that this supposition is false for $n \geqslant 2$. In order to do so, we need to compute the variational symmetry algebra of the Lagrangian (1.27). This can be done by finding the general solution of the variational symmetry condition (1.10), or even more easily (since we know that the variational symmetry algebra is always a subalgebra of the symmetry algebra of the Euler–Lagrange equation), by starting with an arbitrary vector field in the symmetry algebra of equation (1.4) with $k = 2n$, and imposing that it satisfies condition (1.10). The symmetry algebra of the free particle equation has been computed by Lie himself, [1] (see also [3]). When $k = 2$, the symmetry algebra of (1.4) is the eighth Lie algebra in the classification of [8] (see table 1), spanned by the vector fields

$$\partial_x \qquad \partial_u \qquad x\partial_x \qquad x\partial_u \qquad u\partial_x \qquad u\partial_u \qquad x^2\partial_x + xu\partial_u \qquad xu\partial_x + u^2\partial_u. \tag{2.1}$$

This is the well known Lie algebra of the projective group SL(3), which acts locally on $\mathbb{R}^2$ via linear fractional transformations:

$$(x,u) \mapsto \left(\frac{ax+bu+c}{gx+hu+j}, \frac{dx+eu+f}{gx+hu+j}\right) \qquad \det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 1.$$

On the other hand, when $k \geqslant 3$ a basis of the symmetry algebra of (1.4) consists of the $k+4$ vector fields

$$\partial_x \qquad \partial_u \qquad x\partial_x \qquad x\partial_u \qquad u\partial_u \qquad x^2\partial_x + (k-1)xu\partial_u \qquad x^i\partial_u$$
$$2 \leqslant i \leqslant k-1. \tag{2.2}$$

The latter Lie algebra, which is isomorphic to a semidirect product of $\mathfrak{gl}(2)$ with $\mathbb{R}^k$, is the 28th Lie algebra in the classification [8], with $r = k-1$. A straightforward calculation using (2.1) and (2.2) then yields the following proposition:

Table 1. Finite-dimensional Lie algebras of vector fields in $\mathbb{R}^2$. In this table $\mathfrak{h}_2$ stands for the unique two-dimensional solvable non-Abelian Lie algebra. In cases 20–28 we assume that $r \geqslant 1$, in cases 20–21 the functions ξ_i, $1 \leqslant i \leqslant r$, are linearly independent, and in cases 22 and 23 the functions $\eta_i(x)$, $1 \leqslant i \leqslant r$, form a basis of solutions for a linear homogeneous rth-order ordinary differential equation with constant coefficients.

	Generators	Structure
1	$\{\partial_x, \partial_u, \alpha(x\partial_x + u\partial_u) + u\partial_x - x\partial_u\} \quad (\alpha \geqslant 0)$	$\mathbb{R} \ltimes \mathbb{R}^2$
2	$\{\partial_x, x\partial_x + u\partial_u, (x^2-u^2)\partial_x + 2xu\partial_u\}$	$\mathfrak{sl}(2)$
3	$\{u\partial_x - x\partial_u, (1+x^2-u^2)\partial_x + 2xu\partial_u, 2xu\partial_x + (1+u^2-x^2)\partial_u\}$	$\mathfrak{so}(3)$
4	$\{\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u\}$	$\mathbb{R}^2 \ltimes \mathbb{R}^2$
5	$\{\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u\}$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$
6	$\{\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u\}$	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$
7	$\{\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u, (x^2-u^2)\partial_x + 2xu\partial_u, 2xu\partial_x + (u^2-x^2)\partial_u\}$	$\mathfrak{so}(3,1)$
8	$\{\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u\}$	$\mathfrak{sl}(3)$
9	$\{\partial_x\}$	$\mathbb{R}$
10	$\{\partial_x, x\partial_x\}$	$\mathfrak{h}_2$
11	$\{\partial_x, x\partial_x, x^2\partial_x\}$	$\mathfrak{sl}(2)$
12	$\{\partial_x, \partial_u, x\partial_x + \alpha u\partial_u\} \quad (0 < \lvert\alpha\rvert \leqslant 1)$	$\mathbb{R} \ltimes \mathbb{R}^2$
13	$\{\partial_x, \partial_u, x\partial_x, u\partial_u\}$	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$
14	$\{\partial_x, \partial_u, x\partial_x, x^2\partial_x\}$	$\mathfrak{gl}(2)$
15	$\{\partial_x, \partial_u, x\partial_x, u\partial_u, x^2\partial_x\}$	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$
16	$\{\partial_x, \partial_u, x\partial_x, u\partial_u, x^2\partial_x, u^2\partial_u\}$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \approx \mathfrak{so}(2,2)$
17	$\{\partial_x + \partial_u, x\partial_x + u\partial_u, x^2\partial_x + u^2\partial_u\}$	$\mathfrak{sl}(2)$
18	$\{\partial_x, 2x\partial_x + u\partial_u, x^2\partial_x + xu\partial_u\}$	$\mathfrak{sl}(2)$
19	$\{\partial_x, x\partial_x, u\partial_u, x^2\partial_x + xu\partial_u\}$	$\mathfrak{gl}(2)$
20	$\{\partial_u, \xi_1(x)\partial_u, \ldots, \xi_r(x)\partial_u\}$	$\mathbb{R}^{r+1}$
21	$\{\partial_u, u\partial_u, \xi_1(x)\partial_u, \ldots, \xi_r(x)\partial_u\}$	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$
22	$\{\partial_x, \eta_1(x)\partial_u, \ldots, \eta_r(x)\partial_u\}$	$\mathbb{R} \ltimes \mathbb{R}^r$
23	$\{\partial_x, u\partial_u, \eta_1(x)\partial_u, \ldots, \eta_r(x)\partial_u\}$	$\mathbb{R}^2 \ltimes \mathbb{R}^r$
24	$\{\partial_x, \partial_u, x\partial_x + \alpha u\partial_u, x\partial_u, \ldots, x^r\partial_u\}$	$\mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$
25	$\{\partial_x, \partial_u, x\partial_u, \ldots, x^{r-1}\partial_u, x\partial_x + (ru + x^r)\partial_u\}$	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$
26	$\{\partial_x, \partial_u, x\partial_x, x\partial_u, u\partial_u, x^2\partial_u, \ldots, x^r\partial_u\}$	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$
27	$\{\partial_x, \partial_u, 2x\partial_x + ru\partial_u, x\partial_u, x^2\partial_x + rxu\partial_u, x^2\partial_u, \ldots, x^r\partial_u\}$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$
28	$\{\partial_x, \partial_u, x\partial_x, x\partial_u, u\partial_u, x^2\partial_x + rxu\partial_u, x^2\partial_u, \ldots, x^r\partial_u\}$	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$

Proposition 2.1. For all $n \geqslant 1$, the Lie algebra of infinitesimal variational symmetries of the free particle Lagrangian $L = u_n^2$ is spanned by the $n+2$ vector fields

$$\partial_x \qquad \partial_u \qquad 2x\partial_x + (2n-1)u\partial_u \qquad x^i\partial_u \qquad 1 \leqslant i \leqslant n-1. \tag{2.3}$$

In the classification of [8], (2.3) is the 24th Lie algebra (with $\alpha = n - \frac{1}{2}$ and $r = n-1$) when $n \geqslant 2$, and the fourth one (with $\alpha = \frac{1}{2}$) when $n = 1$. To show that the variational symmetry algebra (2.3) of the free particle Lagrangian is not of maximal dimension when $n \geqslant 2$, we are going to construct a Lagrangian $L(x, u, \ldots, u_n)$ whose variational symmetry algebra is of dimension $n+3$ if $n \geqslant 2$. To that end, we just consider a slight generalization of (1.27), namely

$$L = u_n^\alpha. \tag{2.4}$$

For the latter Lagrangian, the variational symmetry condition (1.10) reduces to

$$\alpha\eta^{(n)} + u_n D_x\xi = 0. \tag{2.5}$$

From the prolongation formula (1.14) we easily see that when $n \geqslant 2$ (2.5) is equivalent to the equations

$$\eta^{(n)}|_{u_n=0} = 0 \tag{2.6}$$

and

$$\alpha\eta_u + (1 - n\alpha)\xi_x = [1 - (n+1)\alpha]\xi_u = 0. \tag{2.7}$$

Equation (2.6) is just the condition for the vector field (1.1) to be an infinitesimal symmetry of the free particle equation of order n. Thus, the solutions of (2.5) are just the linear combinations of the vector fields (2.2) which satisfy the additional conditions (2.7). It is now a straightforward matter to check that the general solution of (2.5) depends on $n+2$ arbitrary parameters, except in the following two cases: for $\alpha = 2/(n+1)$ and arbitrary $n \geqslant 2$, and for $n = 2$ and $\alpha = \frac{1}{3}$. In the former case, the solution of (2.5) is

$$\xi = P_2(x) \qquad \eta = \tfrac{1}{2}(n-1)uP_2'(x) + Q_{n-1}(x) \tag{2.8}$$

with P_2 and Q_{n-1} polynomials of degree 2 and $n-1$ in x. In the exceptional case $n = 2$ and $\alpha = \frac{1}{3}$, the general solution of (2.5) depends on $5 = n+3$ parameters, so it just provides an additional example of a second-order Lagrangian with a five-dimensional variational symmetry algebra, inequivalent to (2.8) with $n = 2$. In particular, we have proved the following proposition:

Proposition 2.2. For every $n \geqslant 2$, the Lie algebra of infinitesimal variational symmetries of the Lagrangian

$$L = u_n^{2/(n+1)} \tag{2.9}$$

is spanned by the $n+3$ vector fields

$$\partial_x \qquad \partial_u \qquad 2x\partial_x + (n-1)u\partial_u \qquad x^2\partial_x + (n-1)xu\partial_u \qquad x^i\partial_u$$
$$1 \leqslant i \leqslant n-1. \tag{2.10}$$

The $(n+3)$-dimensional Lie algebra (2.10) is the 27th Lie algebra in the classification [8], with $r = n - 1$. Algebraically, it is a semidirect product of $\mathfrak{sl}(2)$ with $\mathbb{R}^n$.

The previous proposition shows that the free particle Lagrangian does not have a variational symmetry algebra of maximal dimension if $n \geqslant 2$. The question is now whether or not the variational symmetry algebra (2.10) of the Lagrangian (2.4) is of maximal dimension, or in other words whether or not the dimension of the variational symmetry algebra of any Lagrangian of order $n \geqslant 2$ cannot be greater than $n+3$. We shall prove below that this is indeed the case. We shall also deal with the $n = 1$ case, showing that the first-order free particle Lagrangian does have a variational symmetry algebra of maximal dimension, although it shares this property with several other inequivalent first-order Lagrangians.

To prove these results, we shall use the following simple strategy. Let $\mathfrak{g}^{\mathrm{var}}$ be the Lie algebra of real analytic infinitesimal variational symmetries of a Lagrangian $L(x, u, \dots, u_n)$. From [8], we know that there is an open subset $M \subset \mathbb{R}^2$ and appropriate coordinates on M such that $\mathfrak{g}^{\mathrm{var}}|_M$, the restriction to M of the vector fields in $\mathfrak{g}^{\mathrm{var}}$, is one of the 28 types of Lie algebras listed in table 1. Furthermore, since the vector fields in $\mathfrak{g}^{\mathrm{var}}$ are real analytic, it is easy to show that $\mathfrak{g}^{\mathrm{var}}|_M$ and $\mathfrak{g}^{\mathrm{var}}$ have the same dimension (and they are therefore isomorphic). Hence, to find an upper bound on $\dim \mathfrak{g}^{\mathrm{var}}$ we shall simply go through table 1, computing for each Lie algebra $\mathfrak{g}_i$ in the table the most general Lagrangian of order n admitting $\mathfrak{g}_i$ as a Lie algebra of infinitesimal variational symmetries. When performing this calculation, it is convenient to deal separately with the cases $n \geqslant 2$ and $n = 1$. Indeed, if $n \geqslant 2$ we can assume that the dimension d_i of $\mathfrak{g}_i$ is not greater than $n+3$, or equivalently that n satisfies

$$2 \leqslant n \leqslant d_i - 3 \tag{2.11}$$

since for every $n \geqslant 2$ the variational symmetry algebra (2.10) of the Lagrangian (2.9) is $(n+3)$-dimensional. (Were we only interested in finding an upper bound on the dimension of the variational symmetry algebra, without classifying all Lagrangians admitting a variational symmetry algebra of maximal dimension, we could replace the second inequality in (2.11) by a strict inequality.) On the other hand, in the case of first-order Lagrangians we only need to work with Lie algebras of dimension greater than or equal to three, that is

$$d_i \geqslant 3 \qquad (n = 1) \tag{2.12}$$

since the variational symmetry algebra of the first-order free particle Lagrangian is three-dimensional, cf proposition 2.1.

We can further strengthen the first inequality in (2.11), and reduce the number of Lie algebras that we have to consider in the first-order case, by making use of the following proposition:

Proposition 2.3. A non-trivial Lagrangian $L(x, u, \dots, u_n)$ has at most n linearly independent infinitesimal variational symmetries of the form $\eta(x)\partial_u$.

Proof. Indeed, if the $n+1$ vector fields $\eta_i(x)\partial_u$, $1 \leqslant i \leqslant n+1$, are infinitesimal variational symmetries of $L(x, u, \dots, u_n)$, by equations (1.10) and (1.13)–(1.14) we must have

$$\eta_i L_0 + \eta_i' L_1 + \cdots + \eta_i^{(n)} L_n = 0 \qquad 1 \leqslant i \leqslant n+1$$

where

$$L_i = \frac{\partial L}{\partial u_i}. \tag{2.13}$$

Let

$$W[\eta_1, \ldots, \eta_{n+1}] = \det(\eta_i^{(j)})_{\substack{1 \leqslant i \leqslant n+1 \\ 0 \leqslant j \leqslant n}} \tag{2.14}$$

denote the Wronskian of the functions η_i, $1 \leqslant i \leqslant n+1$; if

$$M = \{x \in \mathbb{R} \mid W[\eta_1, \ldots, \eta_{n+1}](x) \neq 0\}$$

then

$$L_i(x, u, \ldots, u_n) = 0 \qquad 0 \leqslant i \leqslant n \tag{2.15}$$

for all $x \in M$ and all $u, u_1, \ldots, u_n$. By the analyticity of the functions η_i, if M is not empty then it is open and dense in $\mathbb{R}$, in which case (2.15) must hold everywhere by continuity. This would imply that L is a function of x only, contradicting the non-triviality assumption. Hence we must have

$$W[\eta_1, \ldots, \eta_{n+1}] = 0$$

everywhere. The following lemma then implies that $\eta_1, \ldots, \eta_{n+1}$ are linearly dependent:

Lemma 2.4.. Let $\eta_i : \mathbb{R} \to \mathbb{R}$ be real analytic for $1 \leqslant i \leqslant k$. If the Wronskian $W[\eta_1, \ldots, \eta_k]$ vanishes for all $x \in \mathbb{R}$, then $\eta_1, \ldots, \eta_k$ are linearly dependent over the reals.

Proof. The proof is an easy induction argument. For $k = 1$, there is nothing to prove. Assuming now that the lemma holds for all positive integers $k \leqslant n$, we shall prove it for $k = n + 1$.

If $W[\eta_1, \ldots, \eta_n]$ vanishes identically, we are done by the induction hypothesis. Otherwise the analyticity of the functions η_i implies that the set

$$M = \{x \in \mathbb{R} \mid W[\eta_1, \ldots, \eta_n](x) \neq 0\} \tag{2.16}$$

is open and dense. From the vanishing of $W[\eta_1, \ldots, \eta_{n+1}]$ it immediately follows that $\eta_1, \ldots, \eta_{n+1}$ satisfy the equation

$$\begin{vmatrix} u & \eta_1 & \cdots & \eta_n \\ u' & \eta_1' & \cdots & \eta_n' \\ \vdots & \vdots & \ddots & \vdots \\ u^{(n)} & \eta_1^{(n)} & \cdots & \eta_n^{(n)} \end{vmatrix} = 0 \tag{2.17}$$

which by (2.16) is a nth-order linear differential equation on the open set M. By the elementary theory of linear differential equations, the $n + 1$ functions η_i must be linearly dependent on M, and since this set is dense in $\mathbb{R}$ the lemma follows.

Remark. Although elementary, this lemma is not totally trivial. For example, it is not true if the functions η_i are merely C^∞.

Let δ_i denote the dimension of the Abelian subalgebra of $\mathfrak{g}_i$ whose elements are the vector fields of the form $\eta(x)\partial_u$ belonging to $\mathfrak{g}_i$. From proposition 2.3, it follows that if $\mathfrak{g}_i$ is a Lie algebra of infinitesimal variational symmetries of a Lagrangian $L(x, u, \ldots, u_n)$ then $\delta_i \leqslant n$. Therefore, we can replace (2.11) by

$$\max(2, \delta_i) \leqslant n \leqslant d_i - 3. \tag{2.18}$$

Similarly, when dealing with first-order Lagrangians we need only consider those Lie algebras in table 1 satisfying

$$\delta_i \leqslant 1 \qquad \text{and} \qquad d_i \geqslant 3. \tag{2.19}$$

We shall now perform the calculation described above, computing for each of the 28 Lie algebras of real analytic vector fields $\mathfrak{g}_i$ listed in table 1, the most general Lagrangian admitting $\mathfrak{g}_i$ as a Lie algebra of variational symmetries. As explained before, we treat separately the cases $n \geqslant 2$ and $n = 1$.

Case I. $n \geqslant 2$.
It turns out that conditions (2.18) are very restrictive, and in fact allow us to completely eliminate many of the 28 types of Lie algebras in table 1 from consideration. Indeed, from (2.18) it follows that any Lie algebra $\mathfrak{g}_i$ such that

$$d_i < \max(5, \delta_i + 3) \tag{2.20}$$

can be safely excluded from our calculation. Thus, the only Lie algebras we have to consider in this case are those of types 5–8, 15–16 and 26–28.

Type 5. In this case, $n = 2$ from (2.18), and an elementary calculation shows that the only second-order Lagrangian admitting this algebra as a Lie algebra of infinitesimal variational symmetries is

$$L = (u_{xx})^{1/3}. \tag{2.21}$$

Type 6. First of all, we can take $n = 3$ by (2.18). Symmetry under the generators ∂_x, ∂_u and $x\partial_u$ implies that L is a function of u_2 and u_3 only. Finally, imposing that $u\partial_u$, $u\partial_x$ and $x\partial_x$ be infinitesimal variational symmetries of $L(u_2, u_3)$ we easily obtain the equations

$$u_2 L_2 + u_3 L_3 = 0$$
$$(u_1 u_3 + 3u_2^2) L_3 = u_1 L$$
$$2u_2 L_2 + 3u_3 L_3 = L$$

from which it immediately follows that L must vanish.

Type 7. Again, (2.18) implies that we can take $n = 3$, and symmetry under the generators ∂_x and ∂_u means that L is a function of the variables (u_1, u_2, u_3). Demanding that the generators $x\partial_x + u\partial_u$, $u\partial_x - x\partial_u$ and $(x^2 - u^2)\partial_x + 2xu\partial_u$ be infinitesimal variational symmetries of L, after some elementary manipulations we arrive at the system

$$\begin{aligned}&(1 + u_1^2) L_1 + 2u_1 u_2 L_2 + (2u_1 u_3 + 3u_2^2) L_3 = 0\\ &u_2 L_2 + 2u_3 L_3 = L\\ &(1 + u_1)^2 L_2 + 6u_1 u_2 L_3 = 0.\end{aligned} \tag{2.22}$$

Solving the last two equations we get

$$L = \Lambda(u_1)v^{1/2} \tag{2.23}$$

with $v = 3u_1u_2 - (1+u_1^2)u_3$.

Substituting (2.23) into the first equation (2.22) yields

$$2(1+u_1^2)\Lambda' + [2u_1 + 3u_2^2(u_1^2-1)v^{-1}]\Lambda = 0$$

which implies that Λ, and hence L, must vanish.

Type 8. As before, from (2.18) and the presence of the generators ∂_x, ∂_u and $x\partial_u$ in $\mathfrak{g}_8$ we deduce that L is a function $L(u_2, \ldots, u_5)$. Combining the equations obtained requiring that the generators $u\partial_u$, $x\partial_x$, $x^2\partial_x + xu\partial_u$ and $u\partial_x$ be infinitesimal variational symmetries of L it is straightforward to obtain the following system of partial differential equations:

$$\begin{pmatrix} u_2 & u_3 & u_4 & u_5 \\ 0 & u_3 & 2u_4 & 3u_5 \\ 0 & 3u_2 & 8u_3 & 15u_4 \\ 0 & 0 & u_2 & 5u_3 \end{pmatrix} \begin{pmatrix} L_2 \\ L_3 \\ L_4 \\ L_5 \end{pmatrix} = \begin{pmatrix} 0 \\ L \\ 0 \\ 0 \end{pmatrix}. \tag{2.24}$$

The general solution of this system is easily found to be

$$L = \frac{c}{u_2}(9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3)^{1/3} \tag{2.25}$$

where c is an arbitrary constant. The Lagrangian (2.25) automatically admits the remaining generator $xu\partial_x + u^2\partial_u$ as an infinitesimal variational symmetry, since we have

$$[u\partial_x - x\partial_u, x^2\partial_x + xu\partial_u] = xu\partial_x + u^2\partial_u.$$

Type 15. An easy calculation along the lines of that for the algebra $\mathfrak{g}_6$ shows that no non-trivial Lagrangian satisfying conditions (2.18) admits this algebra as a Lie algebra of infinitesimal variational symmetries.

Type 16. As above, conditions (2.18) and symmetry under the translations ∂_x and ∂_u allow us to deduce that L is a function $L(u_1, u_2, u_3)$. A calculation totally analogous to that we just performed for $\mathfrak{g}_8$ then shows that the most general Lagrangian of the above form admitting this algebra as a Lie algebra of variational symmetries is

$$L = \frac{c}{u_1}|2u_1u_3 - 3u_2^2|^{1/2} \tag{2.26}$$

with c an arbitrary constant.

Types 26 and 28. In both cases, it is easy to show that there are no non-trivial Lagrangians satisfying the inequalities (2.18) which admit either of these algebras as Lie algebras of infinitesimal variational symmetries. For example, for $\mathfrak{g}_{28}$ conditions (2.18) and symmetry under the generators ∂_x and $x^i\partial_u$, $0 \leqslant i \leqslant r$, imply that L is a function of u_{r+1} and u_{r+2} only. Symmetry under the scalings $x\partial_x$ and $u\partial_u$ is easily seen to imply that

$$u_{r+2}L_{r+2} = L$$

from which it follows that L is of the form

$$L = \lambda(u_{r+1})u_{r+2} = D_x \int^{u_{r+1}} \lambda(s)\,\mathrm{d}s.$$

Hence L is a trivial Lagrangian.

Type 27. In this case, $d_{27} = r+4$ and $\delta_{27} = r+1$, so $n = r+1$ by (2.18). Furthermore, symmetry under the generators ∂_x and $x^i\partial_u$, $0 \leqslant i \leqslant r$, implies in the usual fashion that L is a function of u_n only. Symmetry under the scaling $2x\partial_x + ru\partial_u$ yields

$$L = cu_n^{2/(n+1)} \tag{2.27}$$

with c constant. Finally, it is immediate to check that the latter Lagrangian admits the remaining generator $x^2\partial_x + rxu\partial_u$ as an infinitesimal symmetry. This is in agreement with the results of proposition 2.2.

Remark. Strictly speaking, the above calculations only show that the Lagrangians (2.21), (2.25), (2.26) and (2.27) admit, respectively, the Lie algebras $\mathfrak{g}_5$, $\mathfrak{g}_8$, $\mathfrak{g}_{16}$ and $\mathfrak{g}_{27}$ as *subalgebras* of their variational symmetry algebras. For the Lagrangians (2.21) and (2.27), the calculation preceding proposition 2.2 shows that $\mathfrak{g}_5$ and $\mathfrak{g}_8$ are actually *equal* to the variational symmetry algebras of these Lagrangians. For the remaining two Lagrangians, the same result can be easily deduced from the fact that we have not found any Lie algebra $\mathfrak{g}$ of vector fields on $\mathbb{R}^2$ with the property that there exists a Lagrangian of order n with $2 \leqslant n \leqslant \dim \mathfrak{g} - 4$ admitting $\mathfrak{g}$ as a Lie algebra of variational symmetries.

The results of the above calculation can be conveniently summarized in the following theorem:

Theorem 2.5. If L is a non-trivial Lagrangian of order $n \geqslant 2$, the dimension of its variational symmetry algebra $\mathfrak{g}^{\mathrm{var}}$ is at most $n+3$. Moreover, if $\dim \mathfrak{g}^{\mathrm{var}}$ is exactly equal to $n+3$ then L is equivalent, under an appropriate local change of variables (cf Definition 1.1), to one of the four Lagrangians listed in the following table 2:

Table 2. Lagrangians of order $\geqslant 2$ possessing a variational symmetry algebra of maximal dimension.

Lagrangian	Algebra	Structure
$u_n^{2/(n+1)}$	$\mathfrak{g}_{27}(r = n-1)$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^n$
$\dfrac{1}{u_2}(9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3)^{1/3}$	$\mathfrak{g}_8$	$\mathfrak{sl}(3)$
$\dfrac{1}{u_1}\lvert 2u_1u_3 - 3u_2^2\rvert^{1/2}$	$\mathfrak{g}_{16}$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \approx \mathfrak{so}(2,2)$
$u_2^{1/3}$	$\mathfrak{g}_5$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$

From the previous table we see that, if $n \geqslant 2$ and n is not equal to 2, 3 or 5, then there is one and only one Lagrangian of order n with a variational symmetry algebra of maximal dimension, namely $L = u_n^{2/(n+1)}$ (up to equivalence). On the other hand, for each of the 'anomalous' orders $n = 2, 3, 5$ there are exactly two inequivalent Lagrangians possessing a maximal variational symmetry algebra.

We have shown above that the free particle Lagrangian (1.27) does not possess a variational symmetry algebra of maximal dimension when $n \geqslant 2$. However, since the variational symmetry algebras of two Lagrangians differing by a total derivative do not necessarily coincide, the natural question arises of whether a function $f : J^{n-1} \to \mathbb{R}$ can be found such that the modified free particle Lagrangian

$$L = u_n^2 + D_x f \tag{2.28}$$

admits a variational symmetry algebra of dimension $n+3$ for $n \geqslant 2$. This question can be easily settled in the negative with the help of the previous theorem. Indeed, if (2.28) had a $(n+3)$-dimensional variational symmetry algebra then it would be locally equivalent under a change of variables

$$\bar{x} = \varphi(x, u) \qquad \bar{u} = \psi(x, u) \tag{2.29}$$

to one of the Lagrangians $\bar{L}$ listed in theorem 2.5. If that were the case, according to (1.25) we would have

$$c\bar{L}(\bar{x}, \bar{u}, \ldots, \bar{u}_n) = \frac{u_n^2}{\mathrm{D}_x \varphi} + \frac{\mathrm{D}_x f}{\mathrm{D}_x \varphi} \tag{2.30}$$

where c is a constant and $\bar{u}_i$ is given by equations (1.23) and (1.24). However, it is easy to show that for $n \geqslant 2$ the last component $\bar{u}_n$ of the nth prolongation of (2.29) is linear in u_n, that is we can write

$$\bar{u}_n = A(x, u, \ldots, u_{n-1})u_n + B(x, u, \ldots, u_{n-1}) \tag{2.31}$$

for certain functions $A, B : J^{n-1} \to \mathbb{R}$. (Actually, it can be shown that A depends only on (x, u, u_1), but this will not be needed in what follows.) The (local) equality (2.30) is therefore impossible, since the right-hand side is a second-degree polynomial in u_n, while from (2.31) it follows that the left-hand side is not a polynomial in u_n for any of the Lagrangians of theorem 2.5.

To conclude this case, we shall briefly discuss the connection of the above results with the standard equivalence problem for Lagrangians on the line under point transformations [9], [10], [11]. Kamran and Olver [11], have shown that the latter equivalence problem can be reduced to an $\{e\}$-structure on J^2 for all second-order Lagrangians, except for those of the form

$$L = (A(x, u, u_x)u_{xx} + B(x, u, u_x))^\alpha \qquad \alpha = \tfrac{1}{3} \text{ or } \alpha = \tfrac{2}{3}. \tag{2.32}$$

This result implies that the dimension of the variational symmetry algebra of any second-order Lagrangian not *of the form* (2.32) is at most equal to $\dim J^2 = 4$, whereas a Lagrangian of the form (2.32) could in principle have a symmetry algebra of dimension greater than 4 for appropriate A, B. This is in total agreement with the results of proposition 2.1 and theorem 2.5. For order $n > 2$, the characterization of the Lagrangians for which the equivalence problem reduces to an $\{e\}$-structure on J^n is not precisely known. All we can say in this case is therefore that, since any Lagrangian having the latter property necessarily has a variational symmetry algebra of dimension not greater than $\dim J^n = n+2$, it cannot be equivalent under a point transformation to one of the first three Lagrangians listed in theorem 2.5.

Case II. $n = 1$.
According to condition (2.19), the only Lie algebras of vector fields we have to consider for this case are those of types 1–4, 7, 11–19, 23, and 25. Since we are now dealing exclusively with first-order Lagrangians, there is an additional simplification we can make: if the algebra contains the translations ∂_x and ∂_u, and at least one of the generators $x\partial_x$, $u\partial_x$, $u\partial_u$ or $x\partial_x + u\partial_u$, then any first-order Lagrangian admitting such an algebra as a Lie algebra of variational symmetries can be shown to be necessarily trivial. We can therefore exclude such algebras from consideration, which leaves only the Lie algebras of types 1–3, 11–12, 17–19, and 23 and 25 (for $r = 1$). For each of these algebras, the calculation proceeds exactly along the same lines as in the previous case, except for the following minor point. Since the Lie algebras $\mathfrak{g}_3$ and $\mathfrak{g}_{17}$ do not contain either ∂_x or ∂_u in the canonical coordinates used in reference [8], it is convenient in these cases to change to new (local) coordinates in which one of the generators reduces to a translation. For $\mathfrak{g}_3$, this amounts to a standard change to polar coordinates (see below), whereas for the Lie algebra of type 17, whose generators are

$$\partial_x + \partial_u \qquad x\partial_x + u\partial_u \qquad x^2\partial_x + u^2\partial_u \tag{2.33}$$

the linear change of variables

$$X = x + u \qquad U = x - u$$

transforms the vector fields (2.33) into

$$\partial_X \qquad X\partial_X + U\partial_U \qquad (X^2 + U^2)\partial_X + 2XU\partial_U \tag{2.34}$$

up to unimportant constant factors. In what follows, we shall use these coordinates in preference to those of [8].

By way of example, we shall now give the details of the calculation for the Lie algebra $\mathfrak{g}_3 \approx \mathfrak{so}(3)$, spanned by the vector fields

$$u\partial_x - x\partial_u \qquad (1 + x^2 - u^2)\partial_x + 2xu\partial_u \qquad 2xu\partial_x + (1 + u^2 - x^2)\partial_u.$$

First of all, it is convenient to change to polar coordinates (r, θ), where

$$x = r\cos\theta \qquad u = r\sin\theta$$

and we regard r as the independent variable. Under this change of coordinates, the first two generators transform respectively into

$$\partial_\theta \qquad \cos\theta(1 + r^2)\partial_r + \sin\theta(1 - r^{-1})\partial_\theta$$

up to irrelevant constant factors. The expression for the remaining generator is not needed, since it equals the commutator of the first two, and therefore it is automatically an infinitesimal variational symmetry of L if the first two generators are. Symmetry under the first generator implies that L is a function of r and θ_r only, while imposing symmetry under the second generator we obtain the equation

$$\cos\theta L_r + \frac{1}{r^2}[\sin\theta(1 + r^2\theta_r^2) - r\theta_r\cos\theta]L_{\theta_r} = \left(\theta_r\sin\theta - \frac{2r\cos\theta}{1 + r^2}\right)L.$$

Since L does not depend on θ, equating the coefficients of $\cos\theta$ and $\sin\theta$ in both sides of the previous equation we obtain a system of two linear partial differential equations for L, whose general solution is easily found to be

$$L = c\frac{\sqrt{1 + r^2\theta_r^2}}{1 + r^2} \tag{2.35}$$

c being an arbitrary constant of integration.

The following theorem summarizes the results of similar calculations for the remaining Lie algebras:

Theorem 2.6. Let L be a non-trivial first-order Lagrangian, and let us denote its variational symmetry algebra by $\mathfrak{g}^{\mathrm{var}}$. *Then* $\mathfrak{g}^{\mathrm{var}}$ *is at most three-dimensional, and if* $\dim \mathfrak{g}^{\mathrm{var}}$ *is exactly equal to three then L is equivalent, under an appropriate change of local coordinates, to one of the five Lagrangians listed in the following table 3*:

Table 3. First-order Lagrangians admitting a variational symmetry algebra of maximal dimension.

Lagrangian	Algebra	Structure
$e^{\alpha \arctan u_x}\sqrt{1+u_x^2}$ $(\alpha \geqslant 0)$	$\mathfrak{g}_1$	$\mathbb{R}^2 \ltimes \mathbb{R}$
$\dfrac{\sqrt{1+u_x^2}}{u}$	$\mathfrak{g}_2$	$\mathfrak{sl}(2)$
$\dfrac{\sqrt{1+r^2\theta_r^2}}{1+r^2}$	$\mathfrak{g}_3$	$\mathfrak{so}(3)$
$u_x^{1/(1-\alpha)}$ $(-1 \leqslant \alpha < 1, \alpha \neq 0)$	$\mathfrak{g}_{12}$	$\mathbb{R}^2 \ltimes \mathbb{R}$
$\dfrac{\sqrt{\lvert 1-u_x^2\rvert}}{u}$	$\mathfrak{g}_{17}$	$\mathfrak{sl}(2)$
e^{-u_x}	$\mathfrak{g}_{25}(r=1)$	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R})$

In contrast with the case $n \geqslant 2$, the first-order free particle Lagrangian—the fourth entry in the previous table, for $\alpha = \frac{1}{2}$—possesses a variational symmetry algebra of maximal dimension, but is by no means the only first-order Lagrangian with this property. The second, third and fifth Lagrangians in the previous table, apart from being inequivalent to the free particle Lagrangian in the sense of definition 1.1, generate second-order equations different from the free particle equation.

The above results are in complete agreement with Kamran and Olver's solution of the equivalence problem for first-order Lagrangians on the line under point transformations. Indeed, Kamran and Olver [12], have proved that the latter problem is always reducible to an $\{e\}$-structure on J^1. Hence the dimension of the variational symmetry algebra of a first-order Lagrangian cannot exceed $\dim J^1 = 3$, in agreement with theorem 2.6. Furthermore, Olver, [2], classifies all first-order Lagrangians with a three-dimensional variational symmetry algebra under *complex* point transformations. (Notice, however, that in [2] two Lagrangians differing by a constant factor are *not* considered to be equivalent.) To compare Olver's results with theorem 2.6, it is important to realize that some of the Lagrangians listed in the latter theorem are equivalent under complex point transformations. Indeed, the Lagrangian

$$\bar{L} = e^{\alpha \arctan \bar{u}_{\bar{x}}}\sqrt{1+\bar{u}_{\bar{x}}^2}$$

is equivalent up to a constant factor to

$$L = u_x^{(1-i\alpha)/2} \tag{2.36}$$

under the complex point transformation

$$\bar{x} = x + u \qquad \bar{u} = i(x - u)$$

while the Lagrangians

$$L_1 = \frac{\sqrt{1+r^2\theta_r^2}}{1+r^2} \qquad L_2 = \frac{\sqrt{1+\bar{u}_{\bar{x}}^2}}{\bar{u}}$$

are equivalent up to a constant factor to

$$L = \frac{\sqrt{1-u_x^2}}{u} \tag{2.37}$$

respectively under the complex changes of variables

$$x = \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}(r - r^{-1}) \qquad u = \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}(r + r^{-1})$$

and

$$\bar{x} = x \qquad \bar{u} = \mathrm{i}u.$$

The Lagrangians (2.36), (2.37) and

$$L = \mathrm{e}^{-u_x} \tag{2.38}$$

are not equivalent under complex point transformations, since the complexifications of their variational symmetry algebras are clearly not isomorphic. We therefore conclude that there are only three equivalence classes of first-order Lagrangians with a three-dimensional variational symmetry algebra, up to complex point transformations and multiplication by a constant, given by the Lagrangians (2.36)–(2.38). This is exactly the same result obtained in [2], since the Lagrangian (2.37) is equivalent up to a constant factor to

$$\bar{L} = \sqrt{\bar{u}_{\bar{x}} + \sigma\bar{u}^2}$$

under the point transformation

$$\bar{x} = \frac{1}{\sigma}(u - x) \qquad \bar{u} = \frac{1}{2u}$$

for any complex constant σ.

We conclude this section by pointing out that, if L is one of the Lagrangians listed in theorems 2.5 and 2.6, the functional $\int L\,\mathrm{d}x$ can be naturally interpreted as an arc length invariant under the variational symmetry group of L. Indeed, if $\mathcal{G}$ is a (connected) finite-dimensional Lie group of transformations of the plane, invariance of $\int L\,\mathrm{d}x$ is expressed infinitesimally by (1.17), for every vector field X in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, and is thus equivalent to requiring that $\mathfrak{g}$ be a Lie algebra of infinitesimal variational symmetries of L. Furthermore, if $\mathfrak{g}$ is the variational symmetry algebra of one of the maximally symmetric Lagrangians $L(x, u, \ldots, u_n)$ of theorems 2.5 and 2.6, by the latter theorems the only functionals $\int F(x, u, \ldots, u_r)\,\mathrm{d}x$ of order $r \leqslant n$ invariant under the Lie group of transformations generated by $\mathfrak{g}$ are the constant multiples of $\int L\,\mathrm{d}x$.

The simplest instance of the above remark is provided by the maximally symmetric first-order Lagrangian $\sqrt{1+u_x^2}$, with variational symmetry algebra spanned by the vector fields ∂_x, ∂_u and $u\partial_x - x\partial_u$. The latter is the Lie algebra of the group of Euclidean motions (i.e. translations and rotations) of the plane, and $\sqrt{1+u_x^2}\,\mathrm{d}x$ is of course the well known Euclidean line element. Another interesting first-order example is provided by the Lie algebra $\mathfrak{g}_{12} \approx \mathbb{R} \ltimes \mathbb{R}^2$ for $\alpha = -1$, which is spanned by the vector fields

$$\partial_x \qquad \partial_u \qquad x\partial_x - u\partial_u \tag{2.39}$$

with associated invariant arc length element

$$L = u_x^{1/2}. \tag{2.40}$$

If we perform the change of variables

$$x = t + y \qquad u = t - y \tag{2.41}$$

from (2.39) we obtain the equivalent basis of $\mathfrak{g}$

$$\partial_t \qquad \partial_y \qquad y\partial_t + t\partial_y \tag{2.42}$$

which is the standard basis of the Poincaré group in $\mathbb{R} \times \mathbb{R}$. Under the change of variables (2.41), (2.40) becomes the usual proper time element

$$L = \sqrt{1 - y_t^2} \tag{2.43}$$

which is of course invariant under Poincaré transformations.

For a second-order example, take the maximally symmetric Lagrangian $(u_{xx})^{1/3}$, whose variational symmetry algebra $\mathfrak{g}_5$ (cf theorem 2.5) is the Lie algebra of the group of special affine motions of the plane (generated by translations and linear transformations with determinant equal to one). The integral $\int (u_{xx})^{1/3}\,\mathrm{d}x$ is the standard arc length invariant under the group of special plane affine motions, [13], [14], [15]. Likewise, the Lagrangian $L = u_1^{-1}\sqrt{|2u_1u_3 - 3u_2^2|}$ is symmetric under the Lie algebra $\mathfrak{g}_{16}$, which generates the following realization of SL(2)⊕SL(2) as a Lie group of plane transformations:

$$\bar{x} = \frac{a_1x + b_1}{c_1x + d_1} \qquad \bar{u} = \frac{a_2x + b_2}{c_2x + d_2} \qquad a_id_i - b_ic_i = 1 \qquad i = 1, 2. \tag{2.44}$$

As before, the arc length $\int L\,\mathrm{d}x$ is invariant under the transformations (2.44). In particular, L itself is invariant under the SL(2) subgroup of transformations (2.44) fixing the first coordinate. This should come as no surprise, since L is just the square root of the Schwarzian derivative of u, a well known SL(2) invariant. Finally, a straightforward calculation shows that the Lagrangian

$$u_2^{-1}(9u_2^2u_5 - 45u_2u_3u_4 + 40u_3^3)^{1/3}$$

associated to the Lie algebra $\mathfrak{g}_8 \approx \mathfrak{sl}(3)$ of the group of projective plane transformations, coincides up to a numerical factor with the projective arc element, [16].

3. Divergence symmetries

We shall address in this section the problem of bounding the dimension of the Lie algebra of real analytic infinitesimal divergence symmetries, as well as locally classifying all Lagrangians possessing a divergence symmetry algebra of maximal dimension. We shall follow the same strategy as used in the previous section, with a few minor modifications that we shall explain as we proceed. As in the previous section, we begin by computing the divergence symmetry algebra of the free particle Lagrangian (1.27). To this end, it suffices to write down an arbitrary linear combination of the generators (2.1) or (2.2) (with $k = 2n$) of the symmetry algebra of the free particle equation, and impose that they satisfy the divergence symmetry condition (1.15). The result is given by the following proposition:

Proposition 3.1. For all $n \geqslant 1$, the divergence symmetry algebra of the free particle Lagrangian $L = u_n^2$ is spanned by the $2n+3$ vector fields

$$\partial_x \qquad \partial_u \qquad x\partial_u \qquad 2x\partial_x + (2n-1)u\partial_u \qquad x^2\partial_x + (2n-1)xu\partial_u$$
$$x^i\partial_u \qquad 2 \leqslant i \leqslant 2n-1. \tag{3.1}$$

Notice that the latter Lie algebra is of type 27, with $r = 2n-1$. Proceeding as in the previous section, we should now compute the variational symmetry algebra of the Lagrangian (2.4). However, in this case the latter Lagrangian does not have a divergence symmetry algebra of dimension grater than $n+3$, if $\alpha \neq 2$. This, and the fact that the dimension of the divergence symmetry algebra is bounded by $2n+4$ (the dimension of the symmetry algebra of the Euler–Lagrange equation; cf the introduction), suggests that in this case the free particle Lagrangian has a divergence symmetry algebra of maximal dimension for all $n \geqslant 1$. We shall show below that this is indeed the case. In fact, we shall prove that any Lagrangian with a divergence symmetry algebra of maximal dimension $2n+3$ is locally divergence equivalent to the free particle Lagrangian.

To this end, we need to suitably generalize proposition 2.3 to infinitesimal divergence symmetries. We start by proving the following preliminary lemma:

Lemma 3.2. Suppose that a nth-order Lagrangian $L(x, u, \ldots, u_n)$ satisfies a system of $N > n$ linear partial differential equations of the form

$$\sum_{j=0}^{n} \eta_i^{(j)}(x) L_j = D_x \cdot f_i(x, u, \ldots, u_{n-1}) + \sigma_i(x)u \qquad 1 \leqslant i \leqslant N. \tag{3.2}$$

If the functions $\eta_i(x)$, $1 \leqslant i \leqslant N$, are real analytic and linearly independent, then there exist functions λ_i, $0 \leqslant i \leqslant n$, and μ such that

$$L \approx \frac{1}{2}\sum_{i=0}^{n} \lambda_i(x) u_i^2 + \mu(x)u \tag{3.3}$$

where $\approx$ denotes divergence equivalence (cf Definition 1.2).

Proof. Differentiating (3.2) twice with respect to u_n we obtain

$$\sum_{j=0}^{n} \eta_i^{(j)}(x)\Lambda_j = 0 \qquad 1 \leqslant i \leqslant N$$

where $\Lambda \equiv L_{nn}$. Since $N > n$, arguing as in proposition 2.3 and lemma 2.4 we conclude that

$$\Lambda = L_{nn} = \lambda_n(x),$$

and integrating twice we obtain

$$\begin{aligned} L &= \tfrac{1}{2}\lambda_n(x)u_n^2 + \alpha(x, u, \ldots, u_{n-1})u_n + \beta(x, u, \ldots, u_{n-1}) \\ &\approx \tfrac{1}{2}\lambda_n(x)u_n^2 + a(x, u, \ldots, u_{n-1}) \equiv \hat{L}. \end{aligned} \tag{3.4}$$

Using the identities

$$\left[\frac{\partial}{\partial u}, D_x\right] = 0$$

$$\left[\frac{\partial}{\partial u_j}, D_x\right] = \frac{\partial}{\partial u_{j-1}} \qquad j \geqslant 1$$

it is straightforward to show that a total derivative $D_x \cdot \varphi(x, u, \ldots, u_{n-1})$ *satisfies* (3.2), with $\sigma_i = 0$ and

$$f_i = \sum_{j=0}^{n-1} \eta_i^{(j)} \varphi_j.$$

It follows that $\hat{L}$ also satisfies (3.2), with f_i replaced by a suitable function $\hat{f}_i$, $1 \leqslant i \leqslant N$. For simplicity, we shall write f_i instead of $\hat{f}_i$ in what follows; substituting then (3.4) back into (3.2) and equating the coefficients of u_n on both sides of the resulting equation, we get

$$f_i = \lambda_n(x)\eta_i^{(n)}(x)u_{n-1} + \tilde{f}_i(x, u, \ldots, u_{n-2}) \qquad 1 \leqslant i \leqslant N$$

and

$$\begin{aligned} \sum_{j=0}^{n-1} \eta_i^{(j)}(x)a_j(x, u, \ldots, u_{n-1}) &= (\lambda_n\eta_i^{(n)})'(x)u_{n-1} + D_x \cdot \tilde{f}_i(x, u, \ldots, u_{n-2}) + \sigma_i(x)u \\ &\approx D_x \cdot \tilde{f}_i(x, u, \ldots, u_{n-2}) + [\sigma_i(x) + (-1)^{n-1}(\lambda_n\eta_i^{(n)})^{(n)}(x)]u \qquad 1 \leqslant i \leqslant N. \end{aligned}$$

The lemma then easily follows by induction.

Proposition 3.3. A non-trivial Lagrangian $L(x, u, \ldots, u_n)$ *of order* n *admits at most* $2n$ *linearly independent infinitesimal divergence symmetries of the form* $\eta(x)\partial_u$.

Proof. Indeed, suppose that L had $2n + 1$ linearly independent infinitesimal divergence symmetries $\eta_i(x)\partial_u$, $1 \leqslant i \leqslant 2n+1$. By the divergence symmetry condition (1.15) and the previous lemma (with $\sigma_i = 0$) we obtain

$$L \approx \tilde{L} \equiv \frac{1}{2}\sum_{i=0}^{n} \lambda_i(x)u_i^2 + \mu(x)u \tag{3.5}$$

where $\tilde{L}$ also admits the vector fields $\eta_i(x)\partial_u$, $1 \leqslant i \leqslant 2n+1$, as infinitesimal divergence symmetries by the invariance of this concept under divergence equivalence (cf the introduction). From the divergence symmetry conditions applied to $\tilde{L}$ we then get

$$\sum_{j=0}^{n} \lambda_j \eta_i^{(j)} u_j + \mu \eta_i = D_x f_i \qquad 1 \leqslant i \leqslant 2n+1$$

whence we immediately deduce that

$$\left[\sum_{j=0}^{n} (-1)^j (\lambda_j \eta_i^{(j)})^{(j)}\right] u = D_x g_i \qquad 1 \leqslant i \leqslant 2n+1$$

for certain functions g_i whose explicit expression is not needed. The latter equations can only be satisfied if the coefficient of u in the left-hand side of each of them vanishes, that is we must have

$$\sum_{j=0}^{n} (-1)^j (\lambda_j \eta_i^{(j)})^{(j)} = 0 \qquad 1 \leqslant i \leqslant 2n+1. \tag{3.6}$$

Since L is not trivial by hypothesis, the functions λ_i, $1 \leqslant i \leqslant n$, cannot all vanish identically. Hence there is an open interval M and a positive integer r, $1 \leqslant r \leqslant n$, such that $\lambda_r(x) \neq 0$ for all $x \in M$ and $\lambda_i = 0$ on M for all $i > r$. By (3.6), this implies that the restrictions to M of the functions η_i, $1 \leqslant i \leqslant 2n+1$, are $2n+1$ solutions of a linear differential equation of order $2r \leqslant 2n$ on M. This is of course absurd, since the analyticity of the functions η_i implies that their restrictions to M are still linearly independent.

As in the previous section, we shall now go through the list of inequivalent Lie algebras of vector fields in table 1, computing the most general non-trivial Lagrangian of order n admitting each of these algebras as a Lie algebra of infinitesimal divergence symmetries. Since we are only interested in Lagrangians possessing a divergence symmetry algebra of maximal dimension, by proposition 3.1 we can assume (using the notations of the previous section) that $d_i \geqslant 2n+3$, and from proposition 3.3 we must also have $\delta_i \leqslant 2n$. In other words, we can replace conditions (2.18) and (2.19) by the inequalities

$$\delta_i \leqslant 2n \leqslant d_i - 3. \tag{3.7}$$

Notice that in this case there is nothing special about the order $n = 1$, so we shall just assume that $n \geqslant 1$ throughout. From (3.7), we conclude that we can exclude from our calculation those Lie algebras that do not satisfy the condition

$$d_i \geqslant \max(5, \delta_i + 3) \tag{3.8}$$

thus, we only have to consider the algebras of types 5–8, 15, 16, and 26–28. Another useful fact we shall use in what follows is that, if $\delta_i \geqslant n+1$, then from Lemma 3.2 it follows that L is of the form (3.3); this is the case for the Lie algebras of types 5, 6, and 26–28. Finally, if L is of the form (3.3) and the scaling $u\partial_u$ is an infinitesimal divergence symmetry of L, then we must have

$$\sum_{i=0}^{n} \lambda_i(x) u_i^2 + \mu(x) u = D_x f$$

which can only be satisfied if all the functions λ_i, $0 \leqslant i \leqslant n$, and μ, and therefore L, vanish identically. This allows us to eliminate the Lie algebras of types 6, 26, and 28, and we are therefore left with the algebras of types 5, 7, 8, 15, 16, and 27.

Type 5. To begin with, L is of the form (3.3) in this case. From (3.7), L is first-order, and imposing symmetry under the translations ∂_x and ∂_u we obtain that L must be of the form

$$L = \tfrac{1}{2}\lambda_1 u_x^2 + \mu u$$

with λ_1 and μ constant. Demanding now that $x\partial_x - u\partial_u$ be an infinitesimal divergence symmetry of L we obtain

$$-\tfrac{3}{2}\lambda_1 u_x^2 = D_x f$$

which implies that $\lambda_1 = 0$ and L is trivial.

Type 7. In this case, L is again a first-order Lagrangian by (3.7). Symmetry under the translations ∂_x and ∂_u implies that

$$\frac{\partial L_{11}}{\partial x} = \frac{\partial L_{11}}{\partial u} = 0$$

which after an elementary calculation yields

$$L \approx a(u_x) + bu$$

with b constant. Imposing now symmetry under the scaling $x\partial_x + u\partial_u$ we easily find that $a'' = 0$, and hence L is necessarily trivial.

Type 8. For this algebra, (3.7) implies that we can take $n = 2$. Imposing symmetry under the generators ∂_x, ∂_u, and $x\partial_u$ we easily obtain, as in the previous case, that L is of the form

$$L = a(u_{xx}) + b(x, u, u_x).$$

Imposing now symmetry under the vector field $x\partial_x + 2u\partial_u$ we immediately deduce that $a'' = 0$, and therefore L is actually equivalent to the first order Lagrangian $b(x, u, u_x)$. From this and the fact that $\mathfrak{g}_5$ is a subalgebra of $\mathfrak{g}_8$ we conclude that L is again trivial.

Types 15 and 16. In both of these cases, $n = 1$ by (3.7). The presence of the vector fields ∂_x, ∂_u, and $x\partial_x + u\partial_u$ implies then, as for $\mathfrak{g}_7$, that L is trivial.

Type 27. From (3.7), we deduce that $r = 2n - 1$. We also know that L is of the form (3.3) and demanding symmetry under the translations ∂_x and ∂_u we obtain that L must be of the form

$$L = \frac{1}{2}\sum_{i=1}^{n} \lambda_i u_i^2 + \mu u$$

with λ_i, $1 \leqslant i \leqslant n$, and μ constant. Imposing now that the scaling $2x\partial_x + (2n-1)u\partial_u$ be an infinitesimal divergence symmetry of L we arrive at the equation

$$2\sum_{i=1}^{n-1}(n-i)\lambda_i u_i^2 + (2n+1)\mu u = D_x f \tag{3.9}$$

from which we easily deduce that

$$\mu = \lambda_i = 0 \qquad 1 \leqslant i \leqslant n-1.$$

Therefore

$$L = \tfrac{1}{2}\lambda_n u_n^2 \approx u_n^2.$$

Symmetry under the remaining generators follows from proposition 3.1.

The above results can be summarized in the following theorem:

Theorem 3.4. The divergence symmetry algebra of a non-trivial Lagrangian of order $n \geqslant 1$ is at most $(2n+3)$-dimensional. Moreover, if $L(x,u,\ldots,u_n)$ possesses a $(2n+3)$-dimensional divergence symmetry algebra then L is divergence equivalent (cf Definition 1.2) to the free particle Lagrangian $L = u_n^2$ under an appropriate local change of coordinates.

4. Ordinary differential equations and evolution equations

In this section, we shall explore some elementary applications of the above results on infinitesimal variational symmetries of Lagrangians to the symmetry analysis of ordinary differential equations and evolution equations. For ordinary differntial equations, the basic idea is the following. Suppose that the vector field (1.1) is an infinitesimal variational symmetry of the Lagrangian $L(x,u,\ldots,u_n)$; equation (1.10) is then satisfied, as a consequence of which we have

$$\mathrm{pr}^{(n)} X \cdot L|_{L=0} = 0 \tag{4.1}$$

which is exactly the condition for the vector field X to be an infinitesimal symmetry of the nth-order ordinary differential equation

$$L(x,u,\ldots,u_n) = 0. \tag{4.2}$$

In other words, we have:

Proposition 4.1. Every infinitesimal variational symmetry of a Lagrangian L is an infinitesimal symmetry of the ordinary differential equation $L = 0$.

Of course, the converse need not be true: for instance, the variational symmetry algebra of the free particle Lagrangian $L = u_n^2$ is $(n+2)$-dimensional, whereas the symmetry algebra of the equation $u_n = 0$ is $(n+4)$-dimensional for $n > 2$, eight-dimensional for $n = 2$, and even infinite-dimensional for $n = 1$. From theorem 2.5 we obtain that the ordinary differential equations

$$9u_2^2 u_5 - 45 u_2 u_3 u_4 + 40 u_3^3 = 0 \tag{4.3}$$

and

$$2u_1 u_3 - 3u_2^2 = 0 \tag{4.4}$$

are symmetric respectively under $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. These are famous examples of nth-order ordinary differential equations with an $(n+3)$-dimensional symmetry algebra given by Lie [17] (see also [18]). (In fact, for both of these equations it can be checked that there are no additional infinitesimal symmetries which are not variational symmetries of the corresponding Lagrangians.)

It is a well known fact that a second-order ordinary differential equation with an eight-dimensional symmetry algebra is necessarily equivalent under a local point transformation to the free particle equation $u_{xx} = 0$ (see [2] for a modern proof). For higher-order ordinary differential equations, an analogous result has been proved only for the *linear* case, [18], [19]. It is straightforward, however, to generalize the latter result to arbitrary nonlinear ordinary differential equations by applying the strategy used to prove theorems 2.5 and 2.6.

First of all, by an argument totally analogous to the one used in the proof of proposition 2.3, an nth-order ordinary differential equation has at most n linearly independent infinitesimal symmetries of the form $\eta(x)\partial_u$. By Lie's bound (1.3), the dimension of the symmetry algebra of an ordinary differential equation of order $n \geqslant 3$ is at most $n+4$, which is the dimension of the symmetry algebra of the free particle equation $u_n = 0$. Therefore, to find all ordinary differential equations of order $n \geqslant 3$ with a symmetry algebra of maximal dimension $n+4$, we just have to compute for each algebra $\mathfrak{g}_i$ in table 1 the most general nth-order ordinary differential equation symmetric under $\mathfrak{g}_i$, where the order n satisfies the inequalities

$$\max(3, \delta_i) \leqslant n \leqslant d_i - 4 \tag{4.5}$$

$d_i = \dim \mathfrak{g}_i$ and, as before, δ_i is the dimension of the Abelian subalgebra of $\mathfrak{g}_i$ whose elements are the vector fields of the form $\eta(x)\partial_u$. Equation (4.5) is very restrictive; in fact, it is only satisfied by the Lie algebras of types 8 and 28. For the former of these Lie algebras, an easy calculation shows that there is no fourth-order ordinary differential equation symmetric under it, while for the latter (4.5) implies that $n = r - 1$, and a straightforward computation then proves that the only nth-order ordinary differential equation symmetric under $\mathfrak{g}_{28}$ with $r = n - 1$ is the free particle equation $u_n = 0$. We have thus proved the following theorem:

Theorem 4.2. An nth-order ordinary differential equation $u_n = f(x, u, \ldots, u_{n-1})$ admits a symmetry algebra of maximal dimension $n + 4$ if and only if it is locally equivalent to the free particle equation $u_n = 0$ under a point transformation (1.21).

For evolution equations, the idea behind the proof of proposition 4.1 still applies, but there is an additional complication. Indeed, suppose that the vector field (1.7) is a time-independent time-preserving infinitesimal symmetry of the evolution equation (1.8). The necessary and sufficient condition for this can be expressed as

$$(D_t\eta - u_x D_t\xi)|_{u_t=f} = \mathrm{pr}^{(n)} X \cdot f \tag{4.6}$$

with

$$D_t = \partial_t + u_t\partial_u \tag{4.7}$$

cf [7]. Notice that $\mathrm{pr}^{(n)} X$ in (4.6) can still be computed from (1.13)–(1.14), even if now there is an extra variable t, because it is acting on a function independent of t and of derivatives of u with respect to t. Using (4.7), we can rewrite (4.6) as

$$\mathrm{pr}^{(n)} X \cdot f + f D_x\xi = f \,\mathrm{Div}\, X \tag{4.8}$$

where

$$\mathrm{Div}\, X = \xi_x + \eta_u \tag{4.9}$$

is the divergence of the vector field X with respect to the standard Euclidean measure $\mathrm{d}x\,\mathrm{d}u$. In other words, we have proved the following proposition:

Proposition 4.3. If a vector field (1.1) with zero divergence is an infinitesimal variational symmetry of a Lagrangian $L(x, u, \ldots, u_n)$, then it is a time-independent time-preserving infinitesimal symmetry of the evolution equation $u_t = L(x, u, \ldots, u_n)$.

For example, since all the vector fields in the Lie algebra $\mathfrak{g}_5$ have zero divergence, from the previous proposition and theorem 2.6 it follows that this algebra is a subalgebra of the Lie algebra of time-independent time-preserving infinitesimal symmetries of the evolution equation

$$u_t = (u_{xx})^{1/3}. \tag{4.10}$$

It is important to bear in mind, however, that the connection established by proposition 4.3 between the infinitesimal variational symmetries of a Lagrangian L and the infinitesimal symmetries of its associated evolution equation $u_t = L$ is highly coordinate-dependent, in view of the quite different transformation properties of both objects. On the other hand, the analogous connection between L and the ordinary differential equation $L = 0$ is intrinsic, by equation (1.22).

In general, it would be desirable to remove the restriction that X have zero divergence in proposition 4.3. In particular, we would like to relate each of the maximally symmetric Lagrangians of order $n \geqslant 2$ found in section 2 to an evolution equation, so that the full variational symmetry algebra of each of these Lagrangians be a Lie algebra of time-independent time-preserving infinitesimal symmetries of the associated evolution equation. More precisely, let $\mathfrak{g}$ be a Lie algebra of vector fields in $\mathbb{R}^2$; let

$$\mathcal{E} = \bigcup_{n\geqslant 0} C^\infty(J^n, \mathbb{R})$$

and let $\mathcal{L}[\mathfrak{g}]$ denote the vector space of all Lagrangians admitting $\mathfrak{g}$ as a Lie algebra of variational symmetries. What we want is to find a (not necessarily linear) functional $\mathcal{F} : \mathcal{L}[\mathfrak{g}] \to \mathcal{E}$ such that for every $L \in \mathcal{L}[\mathfrak{g}]$, the evolution equation $u_t = \mathcal{F}[L]$ admits $\mathfrak{g}$ as a Lie algebra of time-independent time-preserving infinitesimal symmetries. Proposition 4.3 tells us that if $\mathfrak{g}$ is a subalgebra of the (infinite-dimensional) Lie algebra of vector fields with zero divergence then we can simply take as $\mathcal{F}$ the canonical injection $\mathcal{L}[\mathfrak{g}] \to \mathcal{E}$, but this choice will not work in general. Instead, we shall try the simple ansatz

$$\mathcal{F}[L] = G(x, u, \ldots, u_k)F(L) \tag{4.11}$$

where the positive integer k and the functions $G : J^k \to \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$ are fixed (i.e. they depend only on $\mathfrak{g}$, but not on $L \in \mathcal{L}[\mathfrak{g}]$). Of course, in principle there is no guarantee that such an ansatz will be appropriate for an arbitrary Lie algebra $\mathfrak{g}$, but we shall now show that if $\mathfrak{g}$ is any of the variational symmetry algebras of maximal dimension of theorem 2.5, the functional (4.11) does in fact do the job.

Indeed, suppose that the evolution equation

$$u_t = G \cdot F(L) \tag{4.12}$$

admits the vector field X given by (1.1) as an infinitesimal symmetry, whenever X is an infinitesimal variational symmetry of L. Using (4.8) and (1.10), we immediately arrive at the equation

$$\mathrm{pr}^{(k)} X \cdot \log G - L\frac{F'}{F}D_x\xi = \eta_u - \xi_u u_x. \tag{4.13}$$

The structure of this equation, which must be valid for all $X \in \mathfrak{g}$ and all $L \in \mathcal{L}[\mathfrak{g}]$ for a given Lie algebra $\mathfrak{g}$, suggests that we set

$$L\frac{F'}{F} = c$$

or equivalently

$$F(L) = L^c \tag{4.14}$$

where c is a constant (depending on $\mathfrak{g}$). Equation (4.13) then simplifies to

$$\mathrm{pr}^{(k)} X \cdot \log G = c D_x \xi + \eta_u - \xi_u u_x \tag{4.15}$$

which is an equation in G only.

The Lie algebras $\mathfrak{g}_8$ and $\mathfrak{g}_{16}$ both contain the vector fields ∂_x, ∂_u, and $x\partial_x + \alpha u \partial_u$ for arbitrary α. Demanding that (4.15) be satisfied by the latter vector fields we deduce that G is a function of the derivatives $(u_1, u_2, \ldots, u_k)$ only, satisfying

$$\sum_{i=1}^{k} (\alpha - i) u_i (\log G)_i = c + \alpha \qquad \forall \alpha \in \mathbb{R}. \tag{4.16}$$

Again, the simplest way of satisfying (4.16) is by assuming that G is a function of one of the derivatives of u only, that is $G = G(u_k)$. In that case, (4.16) immediately yields

$$G = u_k \qquad c = -k. \tag{4.17}$$

Thus, for both $\mathfrak{g}_8$ and $\mathfrak{g}_{16}$ our ansatz for the associated evolution equation is of the form

$$u_t = u_k L^{-k} \tag{4.18}$$

where the positive integer k has yet to be determined. For $\mathfrak{g}_{16}$, imposing that the vector field $x^2\partial_x$ be an infinitesimal symmetry of (4.18) we get, using equation (4.15),

$$k(k-1)\frac{u_{k-1}}{u_k} = 0 \tag{4.19}$$

from which we deduce that $k = 1$ for the Lie algebra $\mathfrak{g}_{16}$. For this value of k, equation (4.15) is automatically satisfied by the remaining generator $u^2\partial_u$. This proves the following proposition:

Proposition 4.4. If the Lagrangian L admits the algebra $\mathfrak{g}_{16}$ *as a Lie algebra of infinitesimal variational symmetries, then the evolution equation*

$$u_t = u_x L^{-1} \tag{4.20}$$

is symmetric under $\mathfrak{g}_{16}$.

For the Lie algebra $\mathfrak{g}_8$, symmetry of (4.18) under the generator $x\partial_u$ implies, by (4.15) and (4.17), that $k > 1$. Imposing now that (4.15) be satisfied by the vector field $x^2\partial_x + xu\partial_u$ we get

$$k(2-k)\frac{u_{k-1}}{u_k} = 0 \tag{4.21}$$

which implies that $k = 2$ for this algebra. It is then straightforward to check that the remaining generators satisfy equation (4.15). Hence we have:

Proposition 4.5. If the Lie algebra $\mathfrak{g}_8$ *is a Lie algebra of infinitesimal variational symmetries for the Lagrangian L, then the evolution equation*

$$u_t = u_{xx} L^{-2} \tag{4.22}$$

is symmetric under $\mathfrak{g}_8$.

Finally, consider the Lie algebra $\mathfrak{g}_{27}$, with $r = n - 1$. Symmetry under the vector fields ∂_x and $x^i \partial_u$, $0 \leqslant i \leqslant n - 1$, implies that G is independent of x, u and derivatives of u of order less than or equal to $n - 1$. If we assume, for simplicity, that G is a function of u_n only, equation (4.15) is the same for the remaining generators $2x\partial_x + (n-1)u\partial_u$ and $x^2\partial_x + (n-1)xu\partial_u$, and implies that

$$G = u_n^{(1-n-2c)/(1+n)}$$

for arbitrary c. Hence:

Proposition 4.6. Let $\mathfrak{g}$ *denote the algebra* $\mathfrak{g}_{27}$ *with* $r = n-1$. *If a Lagrangian L admits* $\mathfrak{g}$ *as a Lie algebra of infinitesimal variational symmetries, then* $\mathfrak{g}$ *is a Lie algebra of infinitesimal symmetries for the evolution equation*

$$u_t = u_n^{(1-n-2c)/(1+n)} L^c \tag{4.23}$$

for all $c \in \mathbb{R}$.

Applying propositions 4.4–4.6 to the maximally symmetric Lagrangians listed in theorem 2.5 and using (4.10), we obtain the following list of nth-order evolution equations possessing an $(n+3)$-dimensional Lie algebra of time-independent time-preserving infinitesimal symmetries, as in table 4.

Table 4. Evolution equations with a Lie algebra of time-independent time-preserving infinitesimal point symmetries of maximal dimension.

Equation	Algebra	Structure
$u_t = (u_{xx})^{13}$	$\mathfrak{g}_5$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$
$u_t = u_x^2 \lvert 2u_x u_{xxx} - 3u_{xx}^2 \rvert^{-1/2}$	$\mathfrak{g}_{16}$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \approx \mathfrak{sl}(2,2)$
$u_t = u_2^3 (9u_2^2 u_5 - 45u_2u_3u_4 + 40u_3^3)^{-2/3}$	$\mathfrak{g}_8$	$\mathfrak{sl}(3)$
$u_t = u_n^{(1-n)/(1+n)}$ $(n \geqslant 2)$	$\mathfrak{g}_{27} (r = n-1)$	$\mathfrak{sl}(2) \ltimes \mathbb{R}^n$

By the result of Sokolov quoted in the introduction, for each of these evolution equations the Lie algebra of vector fields listed next to it in the previous table is exactly equal to its Lie algebra of time-independent time-preserving infinitesimal symmetries. The latter equations thus possess a Lie algebra of time-independent time-preserving infinitesimal (contact) symmetries of maximal dimension. In fact, the above table reproduces the list of equivalence classes (under contact transformations) of evolution equations with a maximal

Lie algebra of time-independent time-preserving infinitesimal contact symmetries found in [6, pp 172–3], see also [20], with the obvious exception of the equation

$$u_t = u_3^4(10u_3^3u_7 - 70u_3^2u_4u_6 - 49u_2^3u_5^2 + 280u_3u_4^2u_5 - 175u_4^4)^{-3/4}$$

the latter equation has proper infinitesimal contact symmetries, not arising—as for the other equations in our table—as the first prolongation of infinitesimal point symmetries. Notice, finally, that equation (4.10) is not listed in [6], since it is equivalent under the *contact* transformation

$$\bar{t} = -t \qquad \bar{x} = u_x^{-1} \qquad \bar{u} = u_x^{-1}u - x$$

to $\bar{u}_{\bar{t}} = (\bar{u}_{\bar{x}\bar{x}})^{-1/3}$, which is the last equation in the previous table for $n = 2$. We have kept both equations in our table, since they are inequivalent under *point* transformations; indeed, the Lie algebras $\mathfrak{g}_5$ and $\mathfrak{g}_{27}$ with $r = 1$, though algebraically isomorphic, are not equivalent under point transformations (the former is primitive and the latter is not, cf [8]).

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